

Quantum correlation exists in any non-product state

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Simultaneous existence of correlation in complementary bases is a fundamental feature of quantum correlation, and we show that this characteristic is present in any non-product bipartite state. We propose a measure via mutually unbiased bases to study this feature of quantum correlation, and compare it with other measures of quantum correlation for several families of bipartite states.

PACS numbers: 03.65.Ud, 03.65.Db, 03.65.Yz.

Quantum systems can be correlated in ways inaccessible to classical objects. This quantum feature of correlations not only is the key to our understanding of quantum world, but also is essential for the powerful applications of quantum information and quantum computation [1–16]. In order to characterize the correlation in quantum state, many approaches have been proposed to reveal different aspects of quantum correlation, such as the various measures of entanglement [6, 7] and the various measures of discord and related measures [14–18], etc. It is believed that some aspects of quantum correlation could still exist without the presence of entanglement and these aspects could be revealed via local measurements with respect to some basis of a local system.

The simultaneous existence of complementary correlations in different bases is revealed very early by the Bell's inequalities [19]. Bell's inequalities quantify quantum correlation via expectation values of local complementary observables. In [20], the feature of genuine quantum correlation is revealed by defining measures based on invariance under a basis change: for a bipartite quantum state, the classical correlation is the maximal correlation present in a certain optimum basis, while the quantum correlation is characterized as a series of residual correlations in the bases mutually unbiased (MU) to the optimum basis. In this paper, we use the fact that the essential feature of the quantum correlation is that it can be present in any two mutually unbiased bases (MUBs) simultaneously. Thus, one of the two bases is not necessarily the optimum basis to reveal the maximal classical correlation in this paper. With respect to the measure proposed here, we shall show that only the product states do not contain quantum correlation. A product state contains neither any quantum correlation nor any classical correlation; while any non-product bipartite state contains correlation that is fundamentally quantum! We shall also reveal interesting properties of this measure by comparing this measure to other measures of quantum correlation for several families of bipartite states.

The MUBs constitute now a basic ingredient in many applications of quantum information processing: quantum state tomography [21], quantum cryptography [22], discrete Wigner function [23], quantum teleportation [24], quantum error correction codes [25], and the mean king's problem [26]. Two orthonormal bases $\{|\psi_i\rangle\}$ and

$\{|\phi_j\rangle\}$ of a d -dimensional Hilbert space H are said to be mutually unbiased if and only if

$$|\langle\psi_i|\phi_j\rangle| = \frac{1}{\sqrt{d}}, \quad \forall 1 \leq i, j \leq d. \quad (1)$$

In a d -dimensional Hilbert space, there exist at least 3 MUBs (when d is a power of a prime number, a full set of $d+1$ MUBs exists, more details can be found in [27]).

We recall the quantity defined in [20]. Let $H_{ab} = H_a \otimes H_b$ with $\dim H_a = d_a$ and $\dim H_b = d_b$ be the state space of the bipartite system A+B shared by Alice and Bob. Let $\{|i\rangle\}$ and $|j'\rangle$ be the orthonormal bases of H_a and H_b respectively. Alice selects a basis $\{|i\rangle\}$ of H_a and performs a measurement projecting her system onto the basis states. The Holevo quantity $\chi\{\rho_{ab}|\{|i\rangle\}\}$ of ρ_{ab} with respect to Alice's local projective measurement onto the basis $\{|i\rangle\langle i|\}$, is defined as $\chi\{\rho_{ab}|\{|i\rangle\}\} = \chi\{p_i; \rho_i^b\} \equiv S(\sum_i p_i \rho_i^b) - \sum_i p_i S(\rho_i^b)$. A basis $\{|i\rangle\}$ that achieves the maximum (denoted as $C_1(\rho_{ab})$) of the Holevo quantity is called a χ -basis of ρ_{ab} . There could exist many χ -bases for a state ρ_{ab} , and the set of these bases is denoted as $\Gamma_{\rho_{ab}}$. Let Ω_{Π^a} be the set of all bases that are mutually unbiased to Π^a , $\Pi^a \in \Gamma_{\rho_{ab}}$. The quantity of quantum correlation in [20], denoted by $Q_2(\rho_{ab})$, is defined as

$$Q_2(\rho_{ab}) \equiv \max_{\Pi^a \in \Gamma_{\rho_{ab}}} \max_{\tilde{\Pi}^a \in \Omega_{\Pi^a}} \chi\{\rho_{ab}|\tilde{\Pi}^a\}. \quad (2)$$

In other words, Q_2 is defined as the Holevo quantity of Bob's accessible information about Alice's results, maximized over Alice's projective measurements in the bases that are mutually unbiased to a χ -basis $\Gamma_{\rho_{ab}}$, and further maximized over all possible χ -bases (if not unique).

Correlation measure based on MUBs.—We now present our approach in a more general way. Let Δ denote the set of all two-MUB sets, i.e.,

$$\Delta = \{\{\{|i_1\rangle\}, \{|j_2\rangle\}\} : \{|i_1\rangle\} \text{ is MU to } \{|j_2\rangle\}\}.$$

We define

$$\mathcal{C}(\rho_{ab}) \equiv \max_{(\Pi_1^a, \Pi_2^a) \in \Delta} \min\{\chi\{\rho_{ab}|\Pi_1^a\}, \chi\{\rho_{ab}|\Pi_2^a\}\}. \quad (3)$$

The quantity \mathcal{C} represents the maximal amount of correlation that is present simultaneously in two MUBs. In a sense, \mathcal{C} is more essential than Q_2 since the maximum in the former one is taken over arbitrarily two

MUBs. Thus, \mathcal{C} may reveal more quantum correlation than Q_2 . Similar to the other usual measures of quantum correlation, \mathcal{C} is local unitary invariant, that is, $\mathcal{C}(\rho_{ab}) = \mathcal{C}(U_a \otimes U_b \rho_{ab} U_a^\dagger \otimes U_b^\dagger)$ for any unitary operators U_a and U_b acting on H_a and H_b respectively.

The nullity of \mathcal{C} .— Now we show that any bipartite quantum state contains nonzero correlation simultaneously in two mutually unbiased bases unless it is a product state, this result is stated as the following theorem.

Theorem. $\mathcal{C}(\rho_{ab}) = 0$ if and only if ρ_{ab} is a product state.

Proof. The ‘if’ part is obvious, and we only need to show the ‘only if’ part. In other words, we only need to prove that $\rho_{ab} = \rho_a \otimes \rho_b$ if either $\chi(\rho_{ab}|\Pi_1^a) = 0$ or $\chi(\rho_{ab}|\Pi_2^a) = 0$ for any MUB pair $(\Pi_1^a, \Pi_2^a) \in \Delta$. It is equivalent to show that both $\chi(\rho_{ab}|\Pi_1^a) \neq 0$ and $\chi(\rho_{ab}|\Pi_2^a) \neq 0$ for a certain MUB pair $(\Pi_1^a, \Pi_2^a) \in \Delta$ if ρ_{ab} is not a product state.

We assume that ρ_{ab} is not a product state, then the maximal classical correlation is nonzero, i.e., $C_1(\rho_{ab}) \neq 0$. Let $\{|e_i\rangle\} \in \Gamma_{\rho_{ab}}$, we have $\chi(\rho_{ab}|\{|e_i\rangle\}) \neq 0$. Therefore, we only need to find a second basis (MU to $\{|e_i\rangle\}$) such that the corresponding Holevo quantity is nonzero. We denote the projective measurement corresponding to $\{|e_i\rangle\}$ by $\Pi = \{\Pi_k = |e_k\rangle\langle e_k|\}$. Then $\Pi(\rho_{ab}) = \sum_k \Pi_k \otimes I_b \rho_{ab} \Pi_k \otimes I_b = \sum_k p_k^{(1)} |e_k\rangle\langle e_k| \otimes \rho_k^{b(1)}$. As $C_1(\rho_{ab}) \neq 0$, we know that $\rho_{k_0}^{b(1)} \neq \rho_b$ and $\rho_{l_0}^{b(1)} \neq \rho_b$ at least for some k_0 and l_0 . We arbitrarily choose a basis $\{|f_i\rangle\}$ that is MU to $\{|e_i\rangle\}$. If $\chi(\rho_{ab}|\{|f_i\rangle\}) \neq 0$, then we already obtain the second basis and the theorem is true.

If $\chi(\rho_{ab}|\{|f_i\rangle\}) = 0$, we can construct the MUB pair as follows. As in this case, the measurement corresponding to $\{|f_i\rangle\}$ yields the following output state

$$\begin{pmatrix} p_1^{(2)} \rho_b & 0 & \cdots & 0 \\ 0 & p_2^{(2)} \rho_b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_d^{(2)} \rho_b \end{pmatrix}.$$

Thus, ρ_{ab} can be represented as

$$\begin{pmatrix} p_1^{(2)} \rho_b & * & \cdots & * \\ * & p_2^{(2)} \rho_b & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & p_d^{(2)} \rho_b \end{pmatrix}$$

with respect to the local basis $\{|f_i\rangle\}$, and at least one of the off-diagonal blocks is not zero (otherwise, ρ_{ab} is a product state). Without loss of generality we assume that the (1,2)-block-entry of the above matrix is nonzero. It follows that there exists a 2 by 2 unitary matrix U_2 , such that, under the local basis $\{U_2 \oplus I_{d-2} |f_i\rangle\}$, the state

admits the form

$$\begin{pmatrix} q_1^{(2)} \varrho_b & * & \cdots & * \\ * & q_2^{(2)} \sigma_b & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & p_d^{(2)} \rho_b \end{pmatrix}$$

with $\varrho_b \neq \rho_b$ and $\sigma_b \neq \rho_b$. That is $\chi\{\rho_{ab}|\{U_2 \oplus I_{d-2} |f_i\rangle\}\} \neq 0$. This unitary matrix U_2 can be chosen as

$$U_2 = \begin{pmatrix} \sqrt{1-\epsilon^2} & \epsilon \\ -\epsilon & \sqrt{1-\epsilon^2} \end{pmatrix}.$$

with ϵ a very small positive number. Even though $\chi\{\rho_{ab}|\{U_2 \oplus I_{d-2} |f_i\rangle\}\}$ could be very small, it is nonzero. As ϵ is a very small and $\chi\{\rho_{ab}|\{|e_i\rangle\}\} \neq 0$, we also have $\chi\{\rho_{ab}|\{U_2 \oplus I_{d-2} |e_i\rangle\}\} \neq 0$. Thus, the Holevo quantity is nonzero at least for a certain MUB pair (i.e., $\{U_2 \oplus I_{d-2} |e_i\rangle\}$ and $\{U_2 \oplus I_{d-2} |f_i\rangle\}$), and therefore $\mathcal{C}(\rho_{ab}) \neq 0$.

Thus, $\mathcal{C}(\rho_{ab}) \neq 0$ for any ρ_{ab} that is not a product state. The proof is completed. ■

In a sense, this theorem implies that, any non-product bipartite state contains genuine quantum correlation, and \mathcal{C} reveals the amount of quantum correlation in the state. In addition, we know that \mathcal{C} is different from the quantity Q_2 in [20] since $Q_2(\rho_{cq}) = 0$ for any classical-quantum state ρ_{cq} while $\mathcal{C} = 0$ only for product states. The difference between the measure \mathcal{C} and other measures of quantum correlation shall be discussed below for several families of bipartite states in more details.

Examples.— Now, we shall calculate the quantity for several families of bipartite states, and see how our measure in terms of MUBs is well justified as a measure of quantum correlation.

For a bipartite pure state with the Schmidt decomposition $|\psi\rangle = \sum_i \sqrt{\lambda_i} |a_i\rangle |b_i\rangle$, $\mathcal{C} = Q_2 = S(\rho_B) = S(\rho_A) = \sum_i -\lambda_i \log_2 \lambda_i$. It can be easily checked that \mathcal{C} coincides with the entropy of either reduced state for any pure state, which is also the usual measure of entanglement in a pure state.

Next, we consider the Werner states of a $d \otimes d$ dimensional system [5],

$$\rho_w = \frac{1}{d(d-\alpha)} (I - \alpha P), \quad (4)$$

where $-1 \leq \alpha \leq 1$, I is the identity operator in the d^2 -dimensional Hilbert space, and $P = \sum_{i,j=1}^d |i\rangle\langle j| \otimes |j\rangle\langle i|$ is the operator that exchanges A and B. For a local measurement with respect to basis states $\{|e_i\rangle\}$ of H_a , with probability $p_i = \frac{1}{d}$, Alice will obtain the k -th basis state $|e_k\rangle$, and Bob will be left with the state $\rho_k^b = \frac{1}{d-\alpha} (I - \alpha |e'_k\rangle\langle e'_k|)$, where $|e'_k\rangle = \sum_j \alpha_{kj} |j'\rangle$ with $\alpha_{kj} = \langle e_k | j \rangle$. It is straightforward to show that

$$\mathcal{C}(\rho_w) = \chi\{p_i; \rho_i^B\} = \log_2\left(\frac{d}{d-\alpha}\right) + \frac{1-\alpha}{d-\alpha} \log_2(1-\alpha). \quad (5)$$

The entanglement of formation E_f for the Werner states is given as $E_f(\rho_w) = h\left(\frac{1}{2}(1 + \sqrt{1 - [\max(0, \frac{d\alpha-1}{d-\alpha})]^2})\right)$, with $h(x) \equiv -x \log_2 x - (1-x) \log_2(1-x)$ [28]. The three different measures of quantum correlation, i.e., \mathcal{C} , the quantum discord D and the entanglement of formation E_f , are illustrated in Fig. 1 for comparison. From this figure, we see that the curve for entanglement of formation intersects the other two curves; thus, E_f can be larger or smaller than \mathcal{C} .

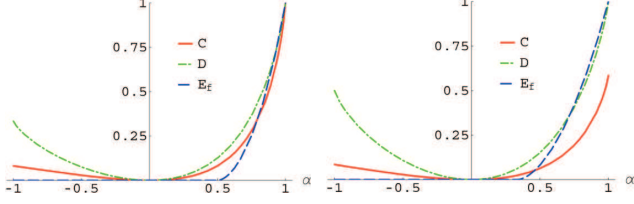


FIG. 1: (color online). Measures of quantum correlation for the Werner states as functions of α when $d = 2$ (left) and $d = 3$ (right). The red curve represents our measure \mathcal{C} , the green curve represents the quantum discord D and the blue curve represents the entanglement of formation E_f .

For the $d \otimes d$ isotropic states

$$\rho = \frac{1}{d^2 - 1}((1 - \beta)I + (d^2\beta - 1)P^+), \quad \beta \in [0, 1], \quad (6)$$

where $P^+ = |\Phi^+\rangle\langle\Phi^+|$, $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle|i'\rangle$ is the maximally entangled pure state in $\mathbb{C}^d \otimes \mathbb{C}^d$. Let $\{|e_k\rangle\langle e_k|\}$ be an arbitrarily given projective measurement on Alice's part. Bob's state after Alice gets the k -th measurement result is

$$\rho_k^b = \frac{1}{d^2 - 1}(d(1 - \beta)I + (d^2\beta - 1)|e'_k\rangle\langle e'_k|),$$

where $|e'_k\rangle = \sum_j \alpha_{kj}|j'\rangle$ with $\alpha_{kj} = \langle e_k|j\rangle$. As the eigenvalues of ρ_k^b does not depend on the basis for Alice's measurement, one can easily show that

$$\begin{aligned} \mathcal{C}(\rho) &= \log_2 d + \frac{d\beta + 1}{d + 1} \log_2 \frac{d\beta + 1}{d + 1} \\ &\quad + \frac{d - d\beta}{d + 1} \log_2 \frac{d - d\beta}{d^2 - 1}. \end{aligned} \quad (7)$$

The entanglement of formation E_f for the isotropic states is given as [29, 30]

$$E_f(\rho) = \begin{cases} 0, & \beta \leq \frac{1}{d}, \\ h(\gamma) + (1 - \gamma) \log_2(d - 1), & \frac{1}{d} < \beta < \frac{4(d-1)}{d^2}, \\ \frac{(\beta-1)d \log_2(d-1)}{d-2} + \log_2 d, & \frac{4(d-1)}{d^2} \leq \beta \leq 1, \end{cases}$$

where $\gamma = \frac{1}{d}(\sqrt{\beta} + \sqrt{(d-1)(1-\beta)})^2$. The quantum discord of the isotropic state is [31]

$$\begin{aligned} D(\rho) &= \beta \log_2 \beta + \frac{1 - \beta}{d + 1} \log_2 \frac{1 - \beta}{d^2 - 1} \\ &\quad - \frac{1 + d\beta}{d + 1} \log_2 \frac{1 - \beta - \frac{1}{d} + d\beta}{d^2 - 1}. \end{aligned}$$

The three different measures of quantum correlation, i.e., \mathcal{C} , the quantum discord D and the entanglement of formation E_f , are illustrated in Fig. 2 for comparison. From this figure, we see that the curve for entanglement of formation intersects the other two curves; thus, E_f can be larger or smaller than \mathcal{C} .

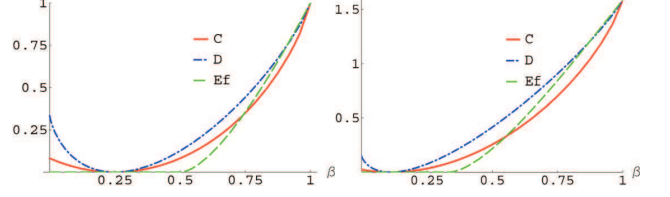


FIG. 2: (color online). Measures of quantum correlation for the isotropic states as functions of β when $d = 2$ (left) and $d = 3$ (right). The red curve represents our measure \mathcal{C} , the green curve represents the quantum discord D and the blue curve represents the entanglement of formation E_f .

As the last example, we consider a family of two-qubit states that are equivalent to Bell-diagonal states under local unitary transformations. This family of states admit the form

$$\sigma_{ab} = \frac{1}{4}(I_2 \otimes I_2 + \sum_{j=1}^3 r_j \sigma_j \otimes \sigma_j). \quad (8)$$

We rearrange the three numbers $\{r_1, r_2, r_3\}$ according to their absolute values and denote the rearranged set as $\{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$ such that $|\bar{r}_1| \geq |\bar{r}_2| \geq |\bar{r}_3|$. Next we show that

$$\mathcal{C}(\sigma_{ab}) = 1 - h\left(\frac{1 + \sqrt{(r_1^2 + r_2^2)/2}}{2}\right). \quad (9)$$

A projective measurement performed on qubit A can be written as $P_{\pm}^a = \frac{1}{2}(I_2 \pm \vec{n} \cdot \vec{\sigma})$, parameterized by the unit vector \vec{n} . When Alice obtains p_{\pm} , Bob will be in the corresponding states $\rho_{\pm}^b = \frac{1}{2}(I_2 \pm \sum_j n_j r_j \sigma_j)$, each occurring with probability $\frac{1}{2}$. The entropy $S(\rho_{\pm}^b)$ reaches its minimum value $h(\frac{1+|\bar{r}_1|}{2})$ when $\vec{n} = (1, 0, 0)$. Let $\vec{n}_1 = (x, y, 0)$ and $\vec{n}_2 = (a, b, 0)$ with $ax + by = 0$, then $P_{\pm}^{(1)}$ is mutually unbiased to $P_{\pm}^{(2)}$, where $P_{\pm}^{(1)} = \frac{1}{2}(I_2 \pm \vec{n}_1 \cdot \vec{\sigma})$, $P_{\pm}^{(2)} = \frac{1}{2}(I_2 \pm \vec{n}_2 \cdot \vec{\sigma})$. It is immediate that $\chi\{\sigma_{ab}|P_{\pm}^{(1)}\} = 1 - h(\frac{1 + \sqrt{x^2 r_1^2 + y^2 r_2^2}}{2})$ and $\chi\{\sigma_{ab}|P_{\pm}^{(2)}\} = 1 - h(\frac{1 + \sqrt{a^2 r_1^2 + b^2 r_2^2}}{2})$. Thus $\mathcal{C}(\sigma_{ab}) = 1 - h(\frac{1 + \sqrt{(r_1^2 + r_2^2)/2}}{2})$ as desired since $h(c)$ is a monotonic decreasing function when $c \geq \frac{1}{2}$. Our quantity \mathcal{C} is compared with the quantum discord D and the entanglement of formation E_f for ρ_1 and ρ_2 in Fig. 3.

From the left figure of Fig. 3, it is clear that \mathcal{C} is quite different from both D and Q_2 . We have $\mathcal{C}(\rho_1) < D(\rho_1)$ when p is closed to $\frac{1}{2}$, while $\mathcal{C}(\rho_1) > D(\rho_1)$ when p is closed to 0 or 1; we also have $\mathcal{C}(\rho_1) = Q_2(\rho_1)$ when $p = \frac{1}{2}$, and $\mathcal{C}(\rho_1)$ increases monotonously while $Q_2(\rho_1)$ decreases

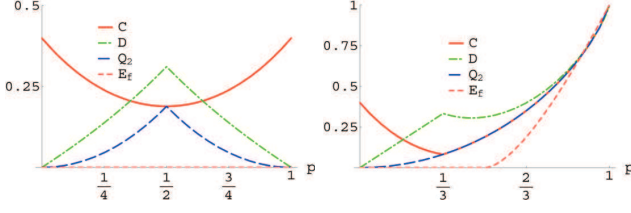


FIG. 3: (color online). Different measures of quantum correlation for two special classes of states: $\rho_1 = \frac{1}{2} |\psi^+\rangle \langle \psi^+| + \frac{p}{2} |\phi^+\rangle \langle \phi^+| + \frac{1-p}{2} |\phi^-\rangle \langle \phi^-|$ (left) and $\rho_2 = p |\psi^-\rangle \langle \psi^-| + \frac{1-p}{2} (|\psi^+\rangle \langle \psi^+| + |\phi^+\rangle \langle \phi^+|)$ (right). In each figure, the red curve represents our measure \mathcal{C} , the green curve represents the quantum discord D , the blue curve represents the measure Q_2 , and the black curve represents the entanglement of formation E_f .

monotonously when p deviates from $\frac{1}{2}$. In Fig. 3, the difference between our measure \mathcal{C} and the other measures is well illustrated by the extreme cases when $p = 0$ or 1 in the left figure and when $p = 0$ in the right figure. For example, for $\sigma = \frac{1}{2} |\psi^+\rangle \langle \psi^+| + \frac{1}{2} |\phi^+\rangle \langle \phi^+|$, our measure has a finite value while the other measures vanish.

Correlation revealed via more MUBs.— In addition, we can define a quantity based on m MUBs ($3 \leq m \leq \dim H_a + 1$), namely,

$$\mathcal{C}_m(\rho_{ab}) \equiv \max_{(\Pi_1^a, \Pi_2^a, \dots, \Pi_m^a) \in \Delta_m} \min\{\chi\{\rho_{ab}|\Pi_1^a\}, \chi\{\rho_{ab}|\Pi_2^a\}, \dots, \chi\{\rho_{ab}|\Pi_m^a\}\}. \quad (10)$$

where

$$\Delta_m = \{(\Pi_1^a, \Pi_2^a, \dots, \Pi_m^a) : \Pi_k^a \text{ is MU to } \Pi_l^a \text{ for any } k \neq l\}.$$

It is clear that $\mathcal{C}_{k+1} \leq \mathcal{C}_k \leq \mathcal{C}$. The following are obvious from the arguments in the previous examples: i) $\mathcal{C}_m(\rho) = 0$ if and only if ρ is a product state, ii) $\mathcal{C}_m = \mathcal{C}$ for both Werner states and the isotropic states, and iii) $\mathcal{C}_3(\sigma_{ab}) = 1 - h(\frac{1+\sqrt{(r_2^2+r_3^2)/2}}{2})$ for the family of two-qubit states in Eq. (8).

In conclusion, we have provided a very different approach to quantify quantum correlation in a bipartite quantum state. Our approach captures the essential feature of quantum correlation: the simultaneous existence of correlations in complementary bases. We have proved that the only states that don't have this feature are the product states, which contains no correlation (classical or quantum) at all. Thus, any non-product state contains correlation that is fundamentally quantum. This feature of quantum correlation characterized here could be the key feature that enables quantum key distribution (QKD) with entangled states, since the quantum correlation that exists simultaneously in MUBs, which can be quantified by \mathcal{C} , is the resource for entanglement-based QKD via MUBs.

Y. Guo is supported by the Natural Science Foundation of China (No. 11301312, No. 11171249), the Natural Science Foundation of Shanxi (No. 2013021001-1, No. 2012011001-2) and the Research start-up fund for Doctors of Shanxi Datong University (No. 2011-B-01). S. Wu is supported by the Natural Science Foundation of China (No. 11275181). This paper is dedicated to professor Jinchuan Hou for his sixtieth birthday.

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